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DETERMINATION OF THE SHAPE OF A CAVITY IN AN ORTHOTROPIC HALF-PLANE GIVEN THE WAVE FIELD AT THE BOUNDARY†

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The shape of a cavity in an elastic orthotropic half-plane, in which vibrations are excited by a point source situated on the surface, is determined from the vertical field of displacements on the surface. The problem is reduced to a system of three non-linear operator equations, and a linearized formulation is proposed for convex cavities. In the final analysis the problem is reduced to solving (1) a system of two boundary integral equations over the boundary of a known cavity, which is done by the boundary element method, and (2) a linear integral equation of the first kind with a smooth kernel. As the problem thus obtained is ill-posed, it is treated by regularization in Tikhonov's sense [1]. The effect of the initial approximation and the vibration frequency on the efficiency of the proposed algorithm is also investigated.

ANALYSIS of problems of this type is usually based on a diffraction formulation [2–5], which requires either the treatment of non-linear operator equations or the minimization of a non-quadratic functional. That approach, however, involves certain difficulties, because of the need to identify imperfections near the surface of the body. Since measuring the field of displacements at the surface provides the most adequate description of the actual imperfection measurement process, the diffraction formulation may be improved by making allowance for the effect of a free boundary on which there are a source and a receiver of vibrations; this will be done in what follows.

1. STATEMENT OF THE PROBLEM

Consider the steady vibrations of an orthotropic elastic half-plane $x_3 \leqslant 0$ with a cavity bounded by a convex smooth curve l. The vibrations are excited by a normal point force of unit strength applied at the origin.

The equations of motion and boundary conditions are (after eliminating the time factor $e^{-i\omega t}$)

$$\begin{aligned} &\sigma_{ij,j} + \rho\omega^2 u_i = 0, \quad i,j = 1,3 \\ &x_3 = 0, \quad \sigma_{13} = 0, \quad \sigma_{33} = -\delta(x_1) \\ &(x_1, x_3) \in l, \quad \sigma_{ij} n_j = 0 \end{aligned} \tag{1.1}$$

where n_j are the components of the unit vector along the normal to l, directed outward relative to the elastic medium.

The components of the stress tensor are related to those of the displacement vector by the generalized Hooke's law

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$$\sigma_{11}=c_{11}u_{1,1}+c_{13}u_{3,3},\quad \sigma_{33}=c_{13}u_{1,1}+c_{33}u_{3,3},\quad \sigma_{13}=c_{55}(u_{1,3}+u_{3,1}) \tag{1.2}$$

where c_{ij} are the elastic constants of the orthotropic material.

We assume that the vertical field of displacements at the surface $x_3=0$ is known, i.e. $u_3(x_1, 0)=f(x_1)$, $x_1\in[a, b]$. The problem is to determine the boundary l of the cavity.

As a first step, we will use a special fundamental solution $U_i^{(m)}(x, \xi)$ for orthotropic half-planes, constructed previously in [6], which satisfies the no-stress condition at the boundary $x_3=0$ and the radiation condition at infinity [7], to obtain a representation of the field of displacements inside the elastic region

$$u_m(\xi)=-\int_l \sigma_{ij}^{(m)}(x,\xi)\,n_j(x)\,u_i(x)\,dl_x+u_m^e(\xi) \tag{1.3}$$

The function $\sigma_{ij}^{(m)}(x, \xi)$ is found from (1.2) with u_i replaced by the fundamental solutions $U_i^{(m)}$; $u_m^e(\xi)$ are the displacements caused by the same surface load in an elastic orthotropic half-plane without imperfections ("reference" displacements).

Using the representation (1.3) and formulae for the limiting values of the integral in (1.3) (see [6]), one reduces the problem to the following system of operator equations

$$\begin{aligned}&\frac{1}{2}u_m(y)=-\int_l \sigma_{ij}^{(m)}(x,y)\,n_j(x)\,u_i(x)\,dl_x+u_m^e(y),\quad y\in l\\&f(\xi')=-\int_l \sigma_{ij}^{(3)}(x,\xi')\,n_j(x)\,u_i(x)\,dl_x+u_3^e(\xi'),\quad \xi'=(x_1,0)\in[a,b]\end{aligned} \tag{1.4}$$

The unknowns in this system are u_i and l. We note that, in the previous analysis of a system of this type in the diffraction formulation, for vibrations of an elastic isotropic plane with a cavity [5], the last equation in (1.4) was replaced by an equation that made it possible to compute the far-field pattern, which was thus supposed to be given.

2. LINEARIZED FORMULATION

To solve system (1.4), we linearize the problem in the neighbourhood of a known state. We know that, in so doing, one has to construct the Fréchet derivative of the non-linear operator generated by system (1.4)—a difficult task owing to the singularities of Eq. (1.4). To avoid this situation, we proceed as follows: consider formula (1.3) for a plane with a cavity l_0 of known shape, slightly different from l and enclosing it; quantities relating to the medium with the cavity l_0 will be marked by a zero superscript

$$u_m^0(\xi)=-\int_{l_0} \sigma_{ij}^{(m)}(x,\xi)\,n_j^0(x)\,u_i^0(x)\,dl_x+u_m^e(\xi) \tag{2.1}$$

Consider the difference

$$u_m(\xi)-u_m^0(\xi)=-\int_l \sigma_{ij}^{(m)}(x,\xi)\,n_j(x)\,u_i(x)\,dl_x+\int_{l_0} \sigma_{ij}^{(m)}(x,\xi)\,n_j^0(x)\,u_i^0(x)\,dl_x \tag{2.2}$$

Transforming the integrals along the contours l and l_0 assumed to be close to each other, we obtain

$$\begin{aligned}&u_m(\xi)-u_m^0(\xi)=-\int_{l_0} \sigma_{ij}^{(m)}(x,\xi)\,(u_i(x)-u_i^0(x))\,n_j^0(x)\,dl_x-\\&-\int_{l_0}(\sigma_{ij}^{(m)}(x,\xi)\,u_i(x))_{,j}\,\nu(x)\,dl_x\end{aligned} \tag{2.3}$$

($\nu(x)$ is a function characterizing the distance between the curves l_0 and l along the inward normal to l_0). Assuming in addition that $u_i(x)=u_i^0(x)$, we obtain from (2.3) a simple version of the linearized problem, in which the first two equations of (1.4) apply to the boundary of the known contour l_0

$$\frac{1}{2}u_m^0(y) = -\int_{l_0} \sigma_{ij}^{(m)}(x, y)n_j^0(x)\, u_i^0(x)\, dl_x + u_m^e(y), \quad m = 1, 3, \quad y \in l_0 \tag{2.4}$$

while the third is an integral equation of the first kind with smooth kernel (since the cavity l_0 lies strictly in the interior of the lower half-plane $x_3 \leqslant 0$, then $\min_{x \in l_0, \xi' \in R_1} |x - \xi'| \geqslant c > 0$)

$$g(\xi') = f(\xi') - u_3^0(\xi') = -\int_{l_0} (\sigma_{ij}^{(3)}(x, \xi')\, u_i^0(x))_{,j}\, \nu(x)\, dl_x, \quad \xi' \in [a, b] \tag{2.5}$$

where $u_3^0(\xi')$ is determined from (2.1).

System (2.4), (2.5) for the three unknown functions $u_i^0(x)$, $\nu(x)$ may be solved successively: one first finds $u_i^0(x)$ from (2.4) and then derives $\nu(x)$ from (2.5).

3. DISCRETIZATION OF THE SYSTEM OF INTEGRAL EQUATIONS

As a first step one uses the boundary element method [8] to solve the singular equations (2.5) approximately. The boundary l_0 is approximated by a polygon with sides (elements) l_q, on each of them $u_i^0(x) = u_{iq}^0$ are assumed to be constant, while the nodal elements u_{iq}^0 are determined by solving a linear algebraic system (a similar discretization method was used in [9] to treat the inverse problem in the diffraction setting)

$$\frac{1}{2}u_{mp}^0 = -\sum_{q=1}^{N} B_{mipq}u_{iq}^0 + u_{mp}^e, \quad p = 1, 2 \ldots N \tag{3.1}$$

$$B_{miq}(\xi) = \int_{l_q} \sigma_{ij}^{(m)}(x, \xi)\, n_{jq}^0 dl_x, \quad B_{mipq} = B_{miq}(y_p), \quad u_{mp}^e = u_m^e(y_p) \tag{3.2}$$

(y_p is the midpoint of l_p). After finding the nodal values of the displacements from (3.1), one can calculate the field inside the elastic half-plane with a cavity l_0, using the following representation

$$u_m^0(\xi) = -\sum_{q=1}^{N} B_{miq}(\xi)\, u_{iq}^0 + u_m^e(\xi) \tag{3.3}$$

and, thanks to the equations of motion (1.1), Eq. (2.5) becomes

$$g(\xi') = \rho\omega^2 \sum_{q=1}^{N} \int_{l_q} U_i^{(3)}(x, \xi')\, u_{iq}^0 \nu(x)\, dl_x \tag{3.4}$$

After suitably parameterizing the qth element $x_1 = x_{1q} + \beta_{1q}t$, $x_3 = x_{3q} + \beta_{3q}t$, $t \in [-1, 1]$, we finally obtain

$$g(\xi') = \rho\omega^2 \int_{-1}^{1} \sum_{q=1}^{N} U_i^{(3)}(x_q + \beta_q t, \xi')\, u_{iq}^0 (\beta_1^2 q + \beta_3^2 q)^{1/2} \nu(x_q + \beta_q t)\, dt, \quad \xi' \in [a, b] \tag{3.5}$$

which is a Fredholm equation of the first kind over a closed interval. The solution of this equation is an ill-posed problem [1]. The solution (3.5) was constructed with the aid of algorithms that provide Tikhonov-type regularizations and used prior information on the solution, e.g. that the unknown function $\nu(x)$ is monotone and non-negative on each element l_q. This is done in practice by linear approximation of $\nu(x)$ on l_q, using regularizing algorithms on sets of well-posedness. Incidentally, the simplest approximation, setting ν_q equal to constants on each element, as proposed for a circular cavity in [9], does not produce good results.

4. NUMERICAL RESULTS

Using the above linearized formulation of the inverse scattering problem and its discretization, we examined the shape of a convex cavity in an orthotropic half-plane. As an example, we took an ellipse with half-axes $d_1=0.2$, $d_2=0.1$, and centre at the point (0, –0.5), for various inclinations to the x_1 axis. We used 8 and 16 boundary elements at frequencies $\kappa=0.1$–0.7 ($\kappa=ka$, $k=\omega(\rho/c_{33})^{1/2}$, $a=\max(d_1, d_2)$). The maximum relative errors in determining the shape of the ellipse at frequencies $\kappa=0.1$, 0.4, 0.7 were 3, 4.8 and 9%, respectively. The drop in the performance of the method as κ increases is due to the need for a finer subdivision of the cavity boundary, non-allowance for the derivatives of the angular displacements u_i^0, and so on. In general, however, this method has proved to be fairly efficient over a wide range of cavity depths and curvatures, especially when one is concerned with such invariant characteristics of imperfections as their areas or the length of the arc l.

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